Reprint ISSN 0973-9424

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND ENGINEERING APPLICATIONS

(IJMSEA)

www.ascent-journals.com

International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 15 No. I (June, 2021), pp. 1-7

ON RAMANUJAN-GÖLLNITZ-GORDON CONTINUED FRACTION

G. VINAY AND K. SHIVASHANKARA
Department of Mathematics, Yuvaraja's College,
University of Mysore, Mysuru-570005, India.
[1] e-mail:: vinaytalakad@gmail.com (Corresponding author)
[2] e-mail: drksshankara@gmail.com

Abstract

In this paper we prove modular relations between $H(q)$ and $H(q^n)$ where $n = 3, 5$ using some modular relations established by S. S. Huang for Göllnitz-Gordon functions.

1. Introduction

Throughout the paper, we let $|q| < 1$ and for positive integer n, we use the standard notation

$$(a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j),$$

Key Words : *Modular equations, Theta functions and Cotinued fraction.*
2010 AMS Subject Classification : 11F20, 33C75 and 11A55.

and

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a;q)_0 = 1.$$

On page 229 of his second notebook [7], Ramanujan recorded the continued fraction

$$H(q) = q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3,q^8)_\infty(q^5,q^8)_\infty} = \frac{q^{1/2}}{1+q_+}\frac{q^2}{1+q^3_+}\frac{a^4}{1+q^5_{+\ldots}}.$$

Without the knowledge of Ramanujan's work, Gӧllnitz [4], Gordon [5], rediscovered and proved the above independently and there by $H(q)$ is called Ramanujan-Gӧllnitz-Gordon continued fraction. Andrews [1], proved the above as a corollary to more general results. In his lost notebook [7], Ramanujan offered an alternative representation of $H(q)$ by

$$H(q) = q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{q^{1/2}}{1_+}\frac{q+q^2}{1_+}\frac{q^4}{1_+}\frac{q^3+q^6}{1_+}\frac{q^8}{1_{+\ldots}}.$$

The above representation of $H(q)$ was first proved by A. Selberg [9], the other proof of above equation have also been given by Andrew [1] and K. G. Ramanathan [8], Ramanujan [7] offered two identities for $H(q)$ namely,

$$\frac{1}{H(q)} - H(q) = \frac{f_4^6}{q^{1/2}f_2^2f_8^4},$$

and

$$\frac{1}{H(q)} + H(q) = \frac{f_2^5}{f_1^2f_4f_8^2}$$

where $f_n \neq (q^n,q^n)_\infty$. H. H. Chan and S. S. Huang [3], have established the relations between $H(q)$ and four continued fraction $H(-q)$, $H(q^2)$, $H(q^3)$ and $H(q^4)$ by employing Ramanujan's theta function found in Chapter 16 [2] is for example the relation between $H(q)$ and $H(q^2)$ is

$$H^2(q) = H(q^2)\left[\frac{1-H(q^2)}{1+H(q^2)}\right].$$

Motivated by this K. R. Vasuki and B. R. Srivatsakumar [10], established three new identities providing the relations between Ramanujan-Gӧllnitz-Gordon continued fraction $H(q)$ and the three continued fraction $H(q^5)$, $H(q^7)$ and $H(q^{11})$.

Two identities analogous to the Rogers-Ramanaujan identities are the so-called Gӧllnitz- Gordon identities [4], [5], given by

$$S(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}, \tag{1.1}$$

and

$$T(q)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q^2;q^2)_n}q^{n^2+2n}=\frac{1}{(q^3,q^8)_\infty(q^4;q_8)_\infty(q^5;q^8)_\infty}.$$

In [6], S. S. Huang established result modular relations between $S(q)$ and $T(q)$. For example

$$S(q^3)S(q)+q^2T(q^3)T(q):=\frac{f_3f_4}{f_1f_{12}}, \tag{1.2}$$

$$S(q^3)T(q)-qS(q)T(q^3):=\frac{f_1f_{12}}{f_3f_4}, \tag{1.3}$$

$$S(q^5)S(q)+q^3T(q^5)T(q):=\frac{f_2f_{10}}{f_1f_{20}}, \tag{1.4}$$

$$S(q^5)T(q)-q^2S(q)T(q^5):=\frac{f_2f_{10}}{f_1f_{20}}. \tag{1.5}$$

In this article, we prove modular relation between $H(q)$ and $H(q^n)$ where $n:=3,5$ by making use of the above Huang identities for $S(q)$ and $T(q)$. We also employing the following in our proof

$$[H^{-1}(q^{1/2})-H(q^{1/2})]^4-16=\frac{f^8(-q)}{qf^8(-q^4)}. \tag{1.6}$$

For a proof, see [12].

2. Proof of Theorem 1 and 2

Theorem 2.1 : If $x:=H(q)$ and $y:=H(q^3)$, then

$$x^4y^3-3x^3y^2+x^3-3x^2y^3+3x^2y-xy^4+3xy^2-y=0. \tag{2.1}$$

Proof : By definition of x and y, we have $x=q^{1/2}\frac{T(q)}{S(q)}$ and $y=q^{3/2}\frac{T(q^3)}{S(q^3)}$.
From (1.2) and (1.3), it follows that

$$\frac{1+xy}{x-y}=\frac{1}{q^{1/2}}\left(\frac{f_4}{f_1}\right)^2\left(\frac{f_3}{f_{12}}\right)^2. \tag{2.2}$$

Raising the power to four on both sides, we obtain

$$\left(\frac{1+xy}{x-y}\right)^4 = \frac{1}{q^4}\left(\frac{f_4}{f_1}\right)^8\left(\frac{f_3}{f_{12}}\right)^8. \tag{2.3}$$

Using (1.6) in the above and then simplifying, we obtain

$$\left(\frac{1+xy}{x-y}\right)^4 = \frac{(H^{-2}(q^{3/2})+H^2(q^{3/2})-2)^2-16}{(H^{-2}(q^{\frac{1}{2}})H^2(q^{1/2})-2)^2-16}. \tag{2.4}$$

Using (1.1) in the above twice, we obtain

$$\left(\frac{1+xy}{x-y}\right)^4 - \frac{\left(\frac{1+y}{y-y^2}+\frac{y-y^2}{1+y}-2\right)^2-16}{\left(\frac{1+x}{x-x^2}+\frac{x-x^2}{1+x}-2\right)^2-16} = 0. \tag{2.5}$$

Factorizing the above equation using maple, we find that

$$A(x,y)B(x,y) = 0,$$

where

$$A(x,y) = x^4y^3 - 3x^3y^2 + x^3 - 3x^2y^3 + 3x^2y - xy^4 + 3xy^2 - y,$$

and

$$\begin{aligned} B(x,y) \;=\; & x^8y^7 - 2x^8y^5 + 7x^7y^6 - 9x^6y^7 + x^5y^8 + x^8y^3 - 15x^7y^4 + 41x^6y^5 - 55x^5y^6 + 15x^4y^7 \\ & -2x^3y^8 + 9x^7y^2 - 55x^6y^3 + 125x^5y^4 - 125x^4y^5 + 41x^3y^6 - 7x^2y^7 + xy^8 \\ & -x^7 + 7x^6y - 41x^5y^2 + 125x^4y^3 - 125x^3y^4 - 55x^2y^5 - 9xy^6 + 2x^5 - 15x^4y \\ & +55x^3y^2 - 41x^2y^3 + 15xy^4 - y^5 - x^3 + 9x^2y - 7xy^2 + 2y^2 - y. \end{aligned}$$

At $q = e^{-\pi/\sqrt{6}}$, from [11] we have

$$x = \sqrt{\frac{\sqrt[4]{2} - \sqrt{\sqrt{6}+\sqrt{3}-\sqrt{2}-2}}{\sqrt[4]{2} + \sqrt{\sqrt{6}+\sqrt{3}-\sqrt{2}-2}}},$$

and

$$y = \sqrt{\frac{\sqrt[4]{2} - \sqrt{2+\sqrt{2}-\sqrt{3}+\sqrt{6}}}{\sqrt[4]{2} + \sqrt{2-\sqrt{2}-\sqrt{3}+\sqrt{6}}}}.$$

Using these in $A(x,y)$ and $B(x,y)$, we find that $A(x,y) = 0$ where $B(x,y) \neq 0$. Thus by identity theorem, we have $A(x,y) = 0$. This completes the proof.

Theorem 2.2 : If $x := H(q)$ and $y := H(q^5)$, then

$$x^5-y+5x^2y-10x^2y^3-10x^3y^4+10x^3y^2-5x^4y^5+10x^4y^3+x^6y^5-5x^5y^2+5xy^4-xy^6=0. \tag{2.6}$$

Proof : By definition of x and y, we have $x = q^{1/2}\frac{T(q)}{S(q)}$ and $y = q^{5/2}\frac{T(q^5)}{S(q^5)}$.
From (1.4) and (1.5), it follows that

$$\frac{1+xy}{x-y} = \frac{1}{q^{1/2}}\left(\frac{f_4}{f_1}\right)\left(\frac{f_3}{f_{12}}\right). \tag{2.7}$$

Raising the power to eight on both sides, we obtain

$$\left(\frac{1+xy}{x-y}\right)^8 = \frac{1}{q^4}\left(\frac{f_4}{f_1}\right)^8\left(\frac{f_3}{f_{12}}\right)^8. \tag{2.8}$$

Using (1.6) in the above and then simplifying, we obtain,

$$\left(\frac{1+xy}{x-y}\right) = \frac{(H^{-2}(q^{5/2})+H^2(q^{5/2})-2)^2-16}{(H^{-2}(q^{1/2})+H^2(q^{1/2})-2)^2-16}. \tag{2.9}$$

Using (1.1) in the above twice, we obtain

$$\left(\frac{1+xy}{x-y}\right) - \frac{\left(\frac{1+y}{y-y^2}+\frac{y-y^2}{1+y}-2\right)^2-16}{\left(\frac{1+x}{x-x^2}+\frac{x-x^2}{1+x}-2\right)-16} = 0. \tag{2.10}$$

Factorizing the above equation using maple, we find that

$$A(x,y)B(x,y) = 0,$$

where

$$\begin{aligned} A(x,y) &= x^5-y+5x^2y-10x^2y^3-10x^3y^4+10x^3y^2-5x^4y^5+10x^4y^3 \\ &\quad +x^6y^5-5x^5y^2+5xy^4-xy^6. \end{aligned}$$

$$\begin{aligned} B(x,y) &= x^{10}y^9-2x^{10}y^7+8x^9y^8-7x^8y^9+x^{10}y^5-11x^9y^6+32x^8y^7-46x^7y^8+3x^6y^9 \\ &\quad +x^5y^{10}-3x^9y^4-3x^8x^5+48x^7y^6-92x^6y^7-3x^5y^8+11x^4y^9-2x^3y^{10} \\ &\quad +7x^9y^2-46x^8y^3+92x^7y^4-28x^6y^5-28x^5y^6-48x^4y^7+32x^3y^8-8x^2y^9+xy^{10} \\ &\quad -9x+8x^8y-32x^7y^2+48x^6y^3+28x^5y^4+28x^4y^5-92x^3y^6+46x^2y^7-7xy^8 \\ &\quad +2xy^8+2x^7-11x^6y+3x^5y^2+92x^4y^3-48x^3y^4+3x^2y^5+3xy^6-x^5-3x^4y \\ &\quad +46x^3y^2-32x^2y^3+11xy^4-y^5+7x^2y-8xy^2+2y^3-y. \end{aligned}$$

At $q - e^{-\pi/2\sqrt{5}}$, from [11] we have

$$x = \sqrt{\sqrt{\sqrt{b}+1} - \sqrt[4]{b}},$$

and

$$y = \sqrt{\sqrt{a}+1} - \sqrt[4]{a},$$

where

$$a = 18 + 8\sqrt{5} + 8\sqrt{2+\sqrt{5}} + 4\sqrt{5+2\sqrt{5}},$$

and

$$b = 18 = 8\sqrt{5} - 8\sqrt{2+\sqrt{5}} - 4\sqrt{5+2\sqrt{5}}.$$

Using this in $A(x,y)$ and $B(x,y)$, we find that $A(x,y) = 0$ where $B(x,y) \neq 0$. Thus by identity theorem, we have $A(x,y) = 0$. This completes the proof.

References

[1] Andrews G. E., An introduction to Ramanujan's "Lost" notebook, Amer. Math. Monthly, 86 (1987), 88-108.

[2] Berndt B. C., Ramanujan notebooks, Part III, Springer-Verlag, New York, (1991).

[3] Chan H. H. and Huang S. S., On the Ramanujan-Göllnitz-Gordon continued fraction, Ramanujan J., 1 (1997), 75-90.

[4] Göllnitz H., Partitionen mit Differenzenbedingungen, J. Reine Angew. Math., 225 (1967), 154- 190.

[5] Gordon B., Some continued fractions of the Rogers-Ramanujan type, Duke Math. J., 32 (1965), 741-748.

[6] Huang S. S., On modular relations for the Göllnitz-Gordon functions with applications to partitions, Journal of Number Theory., 68 (1998), 178-216.

[7] Ramanujan S., Note books (2 volumes). Tata Institute of Fundamental Research. Bombay, (1957).

[8] Ramanathan K. G., Ramanujan's continued fraction, Indian J. Pure Appl. Math., 16 (1985), 71-89.

[9] Selberg A., Uber einige arithmetische Identitaten Avh. Norske vid. -Akad. Oslo I. Mat. -Naturv. Kl., 8 (1936), 3-238.

[10] Srivatsa Kumar B. R., Some studies in the field of special functions in influenced by Ramanujan,Ph.D thesis. Visvesvaraya Technological University, (2008).

[11] Srivatsa Kumar B. R., Evaluations of the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ by modular equations, Indian J. of Mathematics., 48(3) (2006), 275-300.

[12] Sreeramamurthy T. G., A note on explicit evaluations of continued fraction. Advanced Studies in Contempory Mathematics, 9 (2004), 63-80.

[13] Watson G. N., Proof of certain identities in combinatory analysis. J. Indian Math. Proc. Soc., 20:57-69, (1933).